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Modified Wandzura-Wilczek Relation with the Nachtmann Variable

Xiaotong Song*

Institute of Nuclear and Particle Physics

Department of Physics, University of Virginia

P.O.Box 400714, Charlottesville, VA 22904-4714

Abstract

If one retains M^2/Q^2 terms in the kinematics, the Nachtmann variable ξ seems to be more appropriate to describe deep inelastic lepton-nucleon scattering. Up to the first power of M^2/Q^2, a modified Wandzura-Wilczek relation with respect to ξ was derived. Kinematical correction factors are given as functions of ξ and Q^2. A comparison of the modified $g_2^{WW}(\xi)$, and original $g_2^{WW}(x)$ with the most recent g_2 data is shown.

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* Email address: xs3e@virginia.edu

The standard Bjorken variable $x_B = Q^2/2P\cdot q$ is commonly used in the discussion of deep inelastic scattering (DIS). However, if one retains M^2/Q^2 terms in the kinematics another variable ξ introduced by Nachtmann [1] (cf. Greenberg and Bhaumik [2])

$$\xi = 2x/(1 + \sqrt{1 + 4M^2x^2/Q^2}\ ) \tag{1}$$

seems to be more appropriate to describe DIS processes.

For the deep inelastic polarized lepton-nucleon scattering, the asymmetry depends on two spin structure functions g_1 and g_2. The structure function $g_1(x, Q^2)$ can be interpreted as a charge-square weighted quark helicity distribution in the parton model, and the EMC [3] measurement led to a surprising result - the quark spins contribute a very small fraction of the spin of the proton - the so-called " spin puzzle ". Since then many theoretical works and experimental measurements have been done to solve this " puzzle " [4]. Most recently, the second spin structure function g_2, which includes both twist-2 and twist-3 contributions, has been measured with relatively higher precision [5]. Using the operator product expansion (OPE) approach, one can obtain a relation between g_1 and g_2

$$g_2(x, Q^2) = g_2^{WW}(x, Q^2) + \bar{g}_2(x, Q^2), \tag{2}$$

where

$$g_2^{WW}(x, Q^2) \equiv -g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2). \tag{3}$$

If the twist-3 contribution $\bar{g}_2(x, Q^2)$ can be neglected, then Eq.(2) reduces to so called the Wandzura-Wilczek relation. [6,7] An interesting question is that how significant is the twist-3 contribution in g_2. Model predictions (for instance see [8,9]) suggest that the twist-3 contribution is not small compared to g_2^{WW}. The earlier data given by E143 Collaboration [10] and most recent data given by E155 Collaboration [5], however, seem to show that $g_2(x, Q^2)$ is close to $g_2^{WW}(x, Q^2)$ and the twist-3 part of g_2 is rather small.

In an earlier unpublished note [11], we obtained a modified W-W relation (see Eq.(4) below) with respect to the Nachtmann variable ξ. The result was used by E143 Collaboration

[12]. In this brief letter, we present the modified $g_2^{WW}(\xi)$ and compare it with most recent data. More discussions on kinematical corrections arising from the target mass effect are given.

The modified Wandzura-Wilczek relation is

$$g_2^{WW}(\xi, Q^2) = -g_1(\xi, Q^2) + K_2(\xi, Q^2) \int_\xi^1 \frac{dy}{y} \left(\frac{g_1(y, Q^2)}{K_1(y, Q^2)} - 6 \frac{M^2 y^2}{Q^2} \int_y^1 \frac{dz}{z} \frac{g_1(z, Q^2)}{K_1'(z, Q^2)} \right), \quad (4)$$

where the kinematic factors $K_{1,2}$ and K_1' are

$$K_1(y, Q^2) = \frac{1 - M^2 y^2/Q^2}{(1 + M^2 y^2/Q^2)(1 + 3M^2 y^2/Q^2)}, \quad (5a)$$

$$K_2(\xi, Q^2) = \frac{1 - M^2 \xi^2/Q^2}{(1 + M^2 \xi^2/Q^2)^2}, \quad (5b)$$

$$K_1'(z, Q^2) = \frac{1 - M^2 z^2/Q^2}{1 + M^2 z^2/Q^2}. \quad (5c)$$

The derivation of (4) with (5a-c) is given in the appendix. Several remarks are in order.

- (i) In the large-Q^2 limit, all correction factors $K_1(\xi, Q^2)$, $K_1'(\xi, Q^2)$ and $K_2(\xi, Q^2)$ given in (5a), (5b), and (5c) approach unity and (4) becomes

  $$g_2^{WW}(\xi, Q^2) = -g_1(\xi, Q^2) + \int_\xi^1 \frac{dy}{y} \left(g_1(y, Q^2) - 6 \frac{M^2 y^2}{Q^2} \int_y^1 \frac{dz}{z} g_1(z, Q^2) \right) \quad (6)$$

  Considering $M^2/Q^2 \to 0$, and $\xi \to x$, Eq.(6) reduces to the original W-W relation Eq.(3).

- (ii) From Eq.(1), one would expect $\xi_{min} = 0$ and $\xi_{max} = 2/(1 + \sqrt{1 + 4M^2/Q^2}\ )$ for $x = 0 \to 1$. However, since the true momentum fraction carried by quarks is ξ (if quark is massless) rather than x, hence we should take $\xi_{max} = 1$ from the beginning. We note that the derivation of Eq.(4) does not depend on the value of ξ_{max}.

- (iii) To show the correction effect, we first plot K_1, K_2 and K_1' as functions of ξ for $Q^2 = 3$ (GeV)^2, in Fig.1. One can see that for $\xi = 0 \to 1$, $K_1 = 1 \to 0.324$, $K_2 = 1 \to 0.460$ and $K_1' = 1 \to 0.581$. It seems that all correction factors reach the maximum at $\xi = 0$. However, their combined effect presented in Eq.(4) is not so simple.

- (iv) To show some aspects of the correction effect, we assume that g_1 in the integral in Eq.(4) is a constant and define a ratio

$$R(\xi, Q^2) = I(K_1, K_2, \xi, Q^2)/I(1, 1, \xi, Q^2), \tag{7a}$$

where

$$I(K_1, K_2, \xi, Q^2) \equiv K_2(\xi, Q^2) \int_\xi^1 \frac{dy}{y} \left(\frac{1}{K_1(y, Q^2)} - 6\frac{M^2 y^2}{Q^2} \int_y^1 \frac{dz}{z} \frac{1}{K_1'(z, Q^2)} \right). \tag{7b}$$

The ratios $R(\xi, Q^2)$ for $Q^2 = 3$, 5, 10 and 100 (GeV/c)^2 as functions of ξ are shown in Fig.2. One can see that the ratio is quite large at low Q^2 and approaches unity when $Q^2 \to \infty$. However, the ratio (7a) only provides an incomplete information of the kinematical target mass correction to g_2^{WW}. First, the function $g_1(y, Q^2)$ is not a constant but function of y, and secondly, one should take the whole result from Eq.(4), not just the second term.

- (v) As pointed out in [13] that the original Wandzura-Wilczek relation was derived from the Dirac equation for free massless quarks and no higher twist corrections were included. By using the equation of motion with nonzero quark mass and imposing the gauge invariance, an improved Wandzura-Wilczek relation is obtained in [13]

$$g_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) - \frac{m_q}{M} \int_x^1 \frac{dy}{y} \frac{\partial h_T(y)}{\partial y} - \int_x^1 \frac{dy}{y} \Gamma(y), \tag{8}$$

where $h_T(x)$ is the transverse polarization density and $\Gamma(y)$ is related to the multi-parton distribution $h_T(x, x')$. The quark mass-dependent term ($\sim m_q/M$) in (8) is another twist-2 piece in addition to the usual term $g_2^{WW}(x)$. The last term in (8) is a

twist-3 term which is coming from the quark gluon interactions. Assuming the m_q/M term and twist-3 contribution are small, we expect that a modified version of Eq.(8) with the kinematical target mass corrections would be very similar to Eq.(4).

- (vi) Making use of the phenomenologically fitted function to the g_1 data, the modified $g_2^{WW}(\xi)$ in Eq.(4), $g_2^{WW}(\xi)$ in Eq.(6), and the original $g_2^{WW}(x)$ in Eq.(3) are plotted as functions of ξ in Fig.3. The data of $g_2(x)$ are taken from E143 [10] and E155 [5]. From Fig.3, one can see that the effect of the kinematical target mass corrections is rather small relative to the experimental errors. All three g_2^{WW} curves seems to be consistent with the g_2 data. However, more precise data are needed for a significant comparison of g_2^{WW} and g_2. Most recently, two papers [14,15] published on the same topic - target mass corrections on the Wandzura and Wilczek relation - which found that target mass corrections do not affect the W-W relation (2) if all powers in M^2/Q^2 are included. We do not know, however, if this conclusion holds for relation (8). Anyway, since our result (4) holds up to the first power of M^2/Q^2, and target mass corrections have very small effect on the W-W relation, hence there is no contradiction between ours and theirs.

- (vii) It is easy to verify that by changing variable ξ to x and defining $a(x, Q^2) \equiv \sqrt{1 + 4M^2x^2/Q^2} - 1$, Eq.(4) can be rewritten as

$$g_2^{WW}(x, Q^2) = -g_1(x, Q^2) + \frac{1 + a(x, Q^2)/2}{(1 + a(x, Q^2))^2} \int_x^1 \frac{dy}{y} [\frac{1 + 2a(y, Q^2)}{1 + a(y, Q^2)/2} g_1(y, Q^2)$$
$$- \frac{y^2}{(1 + a(y))(1 + a(y, Q^2)/2)^2} \int_y^1 \frac{dz}{z^3} 3a(z)(1 + a(z)/2) g_1(z, Q^2)]. \quad (9)$$

  This is the result obtained in [16].

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Appendix

From Eqs. (47) and (48) in Wandzura's paper [7], for n=2,4,..., we have

$$\int_0^{\xi_{max}} d\xi \cdot \xi^n (1+\frac{M^2\xi^2}{Q^2})[\frac{n}{n+1}(1+\frac{n+2}{n+3}\epsilon)g_1 + (1+\epsilon+\frac{\epsilon^2}{n+4})g_2^{WW}] = 0, \tag{I.1}$$

where

$$\epsilon \equiv \epsilon(\xi, Q^2) \equiv (\frac{2M^2\xi^2}{Q^2})/(1-\frac{M^2\xi^2}{Q^2}).$$

In obtaining Eq.(I.1), g_2 has been replaced by g_2^{WW}, or equivalently, $\bar{g}_2$ has been neglected.

In the large-Q^2 limit, $\epsilon \to 0$, (I.1) becomes

$$\int_0^1 d\xi \cdot \xi^n [\frac{n}{n+1} g_1(\xi, Q^2) + g_2^{WW}(\xi, Q^2)] = 0 \quad (\mathrm{large}-Q^2 \ \mathrm{limit}). \tag{I.2a}$$

From (I.2a), one easily obtains

$$g_2^{WW}(\xi, Q^2) = -g_1(\xi, Q^2) + \int_\xi^1 \frac{dy}{y} g_1(y, Q^2) \quad (\mathrm{large} - Q^2 \ \mathrm{limit}), \tag{I.2b}$$

which is the same form as Eq.(3), but with respect to the variable ξ. Considering $\xi \to x$ in the large-Q^2 limit, the relation (I.2b) approaches the original Wandzura-Wilczek relation Eq.(3).

On the other hand, in the large-n limit, (I.1) becomes

$$\int_0^{\xi_{max}} d\xi \cdot \xi^n (1 + \frac{M^2\xi^2}{Q^2})(1 + \epsilon(\xi, Q^2))[g_1(\xi, Q^2) + g_2^{WW}(\xi, Q^2)] = 0 \quad (\mathrm{large} - n \ \mathrm{limit}). \tag{I.3a}$$

The main contribution to the integral comes from the large-ξ region due to the suppression factor ξ^n. It implies that

$$g_2^{WW}(\xi, Q^2) \simeq -g_1(\xi, Q^2) \qquad (\xi \to \xi_{max}). \tag{I.3b}$$

Considering the large-Q^2 limit (I.2b) and large-n limit (I.3b), it is naturally to assume

$$g_2^{WW}(\xi, Q^2) = -g_1(\xi, Q^2) + K_2(\xi, Q^2) \int_\xi^{\xi_{max}} dy f(y, Q^2) \quad (\mathrm{finite}\ Q^2), \tag{I.4}$$

where $K_2(\xi, Q^2)$ and $f(y, Q^2)$ are two unknown functions to be determined and they have the following behavior in the large-Q^2 limit

$$K_2(\xi, Q^2) \to 1, \qquad f(y, Q^2) \to \frac{g_1(y, Q^2)}{y}. \tag{I.5}$$

Contrast $K_2(\xi, Q^2)$ with $f(y, Q^2)$, the former is a pure kinematical correction factor and does not depend on g_1. Substituting (I.4) into (I.1) and neglecting the ϵ^2 term, we obtain

$$\int_0^{\xi_{max}} d\xi \cdot \xi^n (1 + \frac{M^2\xi^2}{Q^2})[(1 + \frac{2n+3}{n+3}\epsilon)\frac{-g_1}{n+1} + (1+\epsilon)K_2(\xi, Q^2) \int_\xi^{\xi_{max}} dy f(y, Q^2)] = 0. \tag{I.6}$$

Unlike the derivation of (I.2b) from (I.2a), we have to use *one* equation, (I.6), to determine *two* unknown functions. Since K_2 is a pure kinematical correction factor satisfies the large-Q^2 behavior (I.5), we may choose

$$K_2(\xi, Q^2) \equiv (1 + \epsilon(\xi, Q^2))^{-1}(1 + \frac{M^2\xi^2}{Q^2})^{-1}, \tag{I.7}$$

and rewrite (I.6) as

$$\int_0^{\xi_{max}} \frac{d\xi}{n+1} \cdot \xi^{n+1}[f(\xi, Q^2) - (1 + \frac{M^2\xi^2}{Q^2})(1 + \frac{2n+3}{n+3}\epsilon)\frac{g_1(\xi, Q^2)}{\xi}] = 0, \tag{I.8}$$

where we have exchanged the order of the integrals in the second term of Eq.(I.6).

To determine the second unknown function $f(\xi, Q^2)$, we decompose it into two pieces

$$f(\xi, Q^2) = f^{(0)}(\xi, Q^2) + f^{(1)}(\xi, Q^2), \tag{I.9}$$

where $f^{(0)}(\xi, Q^2) \sim O(1)$ and $f^{(1)}(\xi, Q^2) \sim O(M^2/Q^2)$ is a small term. Since $f^{(1)}(\xi, Q^2) \to 0$ in the limit $Q^2 \to \infty$, the function $f^{(0)}(\xi, Q^2)$ must satisfies large-Q^2 behavior (I.5). We purposely choose

$$f^{(0)}(\xi, Q^2) = (1 + \frac{M^2\xi^2}{Q^2})(1 + 2\epsilon)\frac{g_1(\xi, Q^2)}{\xi}. \tag{I.10}$$

From (I.8), (I.9), and (I.10), we have

$$\int_0^{\xi_{max}} \frac{d\xi}{n+1} \cdot \xi^{n+1}[f^{(1)}(\xi, Q^2) + (1 + \frac{M^2\xi^2}{Q^2})\frac{3\epsilon(\xi, Q^2)}{n+3}\frac{g_1(\xi, Q^2)}{\xi}] = 0. \tag{I.11}$$

To determine small unknown function $f^{(1)}(\xi, Q^2)$, we put

$$f^{(1)}(\xi, Q^2) \equiv \xi \int_\xi^{\xi_{max}} dy \eta(y, Q^2), \tag{I.12}$$

where $\eta(y, Q^2)$ should be the order of $O(M^2/Q^2)$. Substituting (I.12) into (I.11), one obtains

$$\int_0^{\xi_{max}} d\xi \frac{\xi^{n+3}}{(n+1)(n+3)}[\eta(\xi, Q^2) + 3\epsilon(\xi, Q^2)(1 + \frac{M^2\xi^2}{Q^2})\frac{g_1(\xi, Q^2)}{\xi^3}] = 0. \tag{I.13}$$

This equation can be satisfied for all n (n=2,4,...) only if the term in the bracket vanishes. Therefore

$$\eta(\xi, Q^2) = -3\epsilon(\xi, Q^2)(1 + \frac{M^2\xi^2}{Q^2})\frac{g_1(\xi, Q^2)}{\xi^3}, \tag{I.14}$$

which is indeed the order of $O(M^2/Q^2)$. Substituting (I.9), (I.10), (I.12), and (I.14) into (I.4), we finally obtain the modified W-W relation Eq.(4) with correction factors (5a), (5b), and (5c).

FIGURES

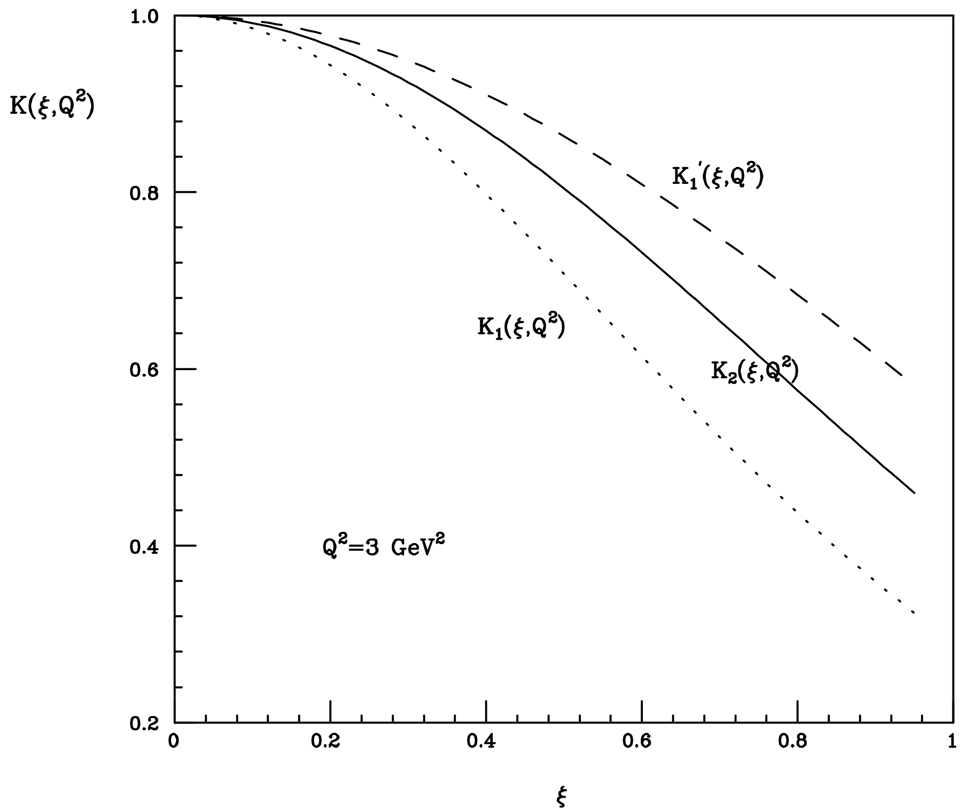

Fig. 1

FIG. 1. The kinematical correction factors (see Eqs.(5a), (5b), and (5c)) plotted as functions of the Nachtmann variable ξ at Q^2=3 (GeV/c)^2.

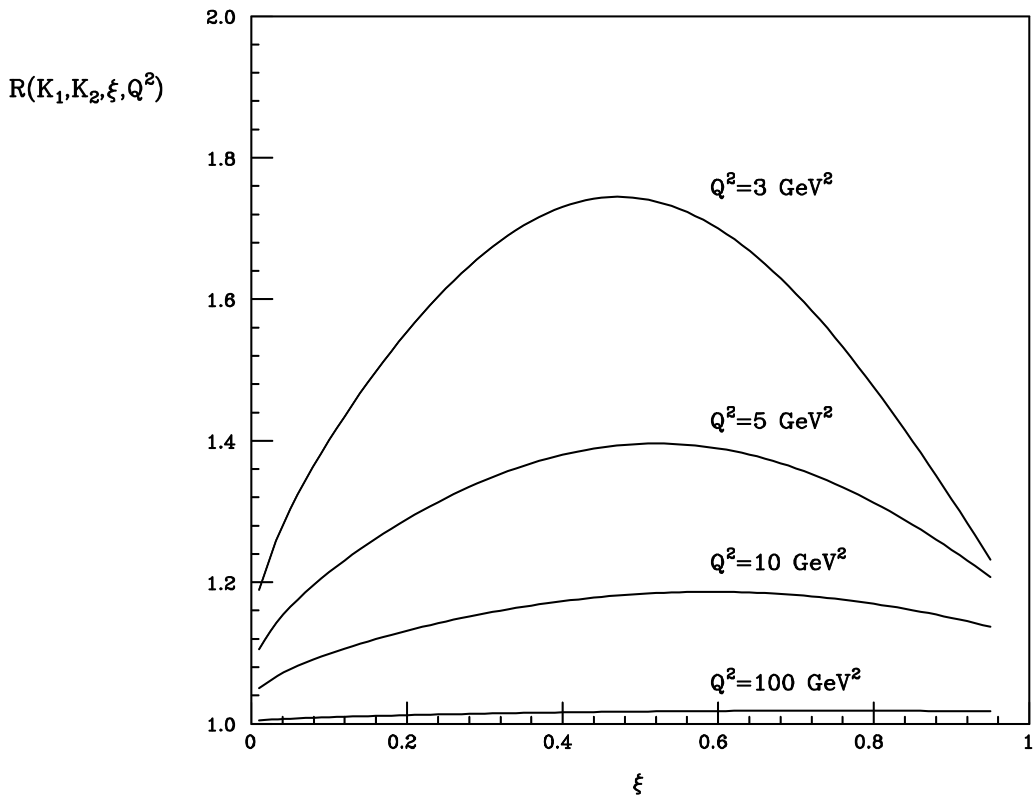

Fig. 2

FIG. 2. The correction ratios (see Eq.(6a,b)), plotted as functions ξ, for Q^2=3, 5, 10 and 100 $(\mathrm{GeV/c})^2$.

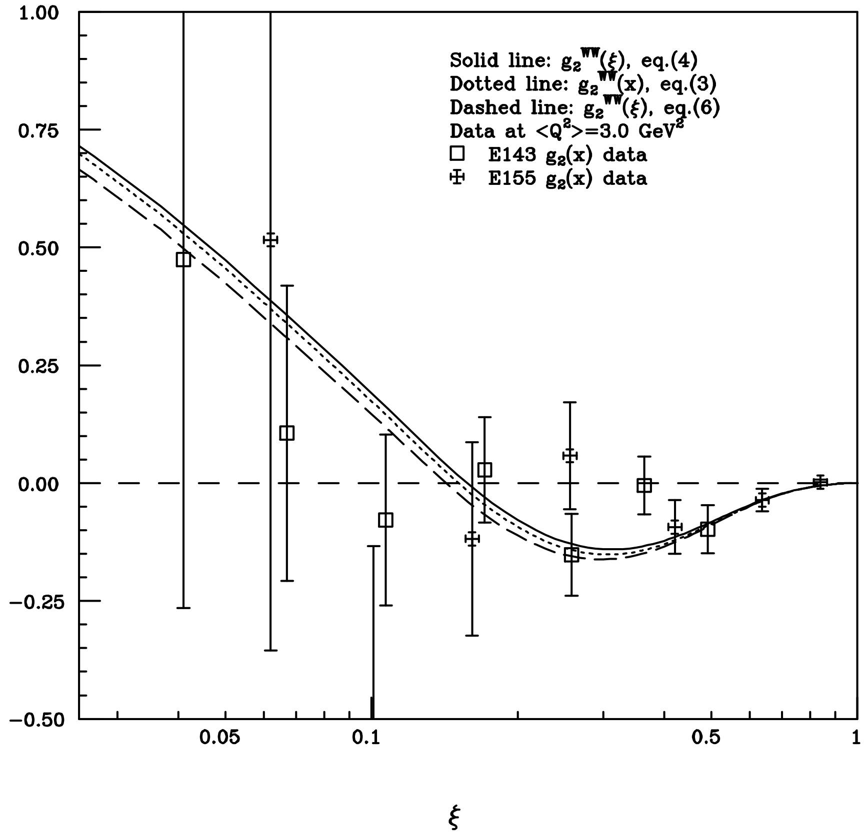

FIG. 3. Modified $g_2^{WW}(\xi)$, Eq.(4) and Eq.(6), and the original W-W relation, Eq.(3) plotted as functions ξ, for Q^2=3 (GeV/c)^2. Data are taken from E143 and E155.